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Effect of topographic convergence on erosion processes

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Abstract. We consider a basin made of an impermeable soil, eroded by water flow (not by weathering). Our approach is based on standard deterministic models, within a 'streamlet' picture where the flow is always directed downhill. The central assumption is that erosion occurs only when the surface shear stress τ (due to water flow) is above a certain threshold τ_c. Some features of the landscape are simplified by the existence of τ_c, and do not depend on detailed assumptions on the behaviour above τ_c. In particular, if the initial landscape is a U-shaped valley, we can construct the 'line of attack'—i.e. the border of the eroded regions—by a simple prescription.

1. Principles

The oral version of this paper covered both problems of dry sand and wet sand. For the written version, in view of the restrictions on size, we select only one topic related to wet systems: namely erosion processes.

A deterministic description of water basins implies a detailed knowledge of many erosion and sedimentation processes [1, 2]. Erosion may be due to weathering conditions (e.g. the impact of rain droplets) [3] or to tear-off by surface flow: here we focus on the latter case. Early mechanical models [4] postulated a local sediment flux F which would be uniquely determined by the local slope $|\nabla z| = \theta$ and the local water flux Q. This is somewhat oversimplified, since F is a sum from upstream contributions which occurred at different slopes and smaller fluxes. Another important feature, particularly emphasized in [5], is that erosion occurs only when the tangential stress due to flow,

$$\tau = \rho g h \theta \tag{1}$$

is above a certain threshold τ_c. In (1) ρ is the water density, g is the gravitational acceleration and h is the local water thickness.

Our aim in the present article is to emphasize two facts.

(1) The existence of a threshold stress is enough to fix the 'line of attack', i.e. the border of the eroded regions after the onset of rain on a given landscape. We show this in section 2, with two landscapes (an inclined plane and a tilted U-shaped valley) where topographic convergence is important. The concept of a line of attack is well known. It provides one answer to the classical question: '*where do channels begin*?' raised originally by Montgomery and Dietrich [6, 7].

(2) Consider now a mature basin, experiencing erosion, sedimentation (plus a slight uplift velocity U allowing for steady-state regimes). We show in section 3 that there is a good surprise: at small U, the flow velocity in the eroded regions remains always very close to the threshold value V_c (associated with τ_c). It is then possible to predict some universal

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features of the profiles, which are not sensitive to the details of the erosion process beyond threshold.

Points (1) and (2) are simple illustrations of the ideas described in [5], but they do bring some simplification. Let us list here the main assumptions involved.

(a) The solid is impermeable: rain flows only at the surface, with a local thickness h, a local velocity $\boldsymbol{V}$ and the resulting local flux $\boldsymbol{Q} = \boldsymbol{V}h$.
(b) The material is not sensitive directly to droplet impacts, but is eroded only by surface flows, when $\tau > \tau_c$.
(c) The hydrodynamic flow is turbulent, and is locally in a steady state with the following relation between velocity and slope $|\nabla z| = \theta$:

$$V^2 = kgh\theta \qquad (2)$$

where k is a numerical constant. Of course, (2) assumes that the slopes vary very smoothly, i.e. that curvature effects are negligible.
(d) The flow velocity points downhill. This neglects certain lateral exchanges between 'streamlets', which are weak (see the appendix). We do assume that there is indeed a well defined downhill direction at all points. Mathematically, this corresponds to $\theta \neq 0$ everywhere. Physically, this means that our landscapes do not have any lake-forming regions.
(e) The critical stress τ_c is dependent only on V, not on the slope. If we accept (1) and (2) the local stress is $\tau = k^{-1}\rho V^2$. Then the threshold stress τ_c corresponds to a fixed threshold velocity V_c:

$$\rho V_c^2 = k\tau_c. \qquad (3)$$

For laboratory experiments, we would need rather small values of V_c (say 10 cm s^{-1}) and this implies very weak solids ($\tau_c \sim 10^{-4}$ atm $= 10^2$ dynes cm^{-2}).

2. Flows below threshold and the line of attack

The main feature of our systems, emphasized in [5], is the convergence of streamlets, as indicated in figure 1. Consider an element dl along an isolevel line. This collects the water from an upstream area dA and must have

$$Vh\,\mathrm{d}l = p\,\mathrm{d}A \qquad (4)$$

where p is the rainfall per unit area and unit time.

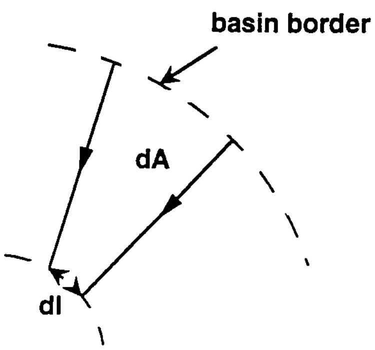

Figure 1. Definition of the collection factor dA/dl.

If we compare this to (2), we arrive at the basic relation

$$V^3 = kg\theta p \frac{\mathrm{d}A}{\mathrm{d}l}. \tag{5}$$

We call $\mathrm{d}A/\mathrm{d}l$ the collection factor. The role of the collection factor was emphasized long ago [6].

We shall now illustrate (5) with two examples.

2.1. *Inclined plane*

Here the slope is constant $\theta = \theta_0$ and the collection factor is simply equal to the distance x from the crest line. We may thus rewrite (5) in the form

$$\left(\frac{V}{V_c}\right)^3 = \frac{x}{L} \tag{6}$$

where L is a characteristic length

$$L = \frac{V_c^3}{kg\theta_0 p}. \tag{7}$$

The top part of the landscape ($0 < x < L$) is not eroded. The line of attack corresponds to $x = L$. The length L is inversely proportional to the rainfall p and to the slope θ_0. It is also proportional to $V_c^3 \sim (\tau_c)^{3/2}$.

For a laboratory experiment, with $V_c \sim 10$ cm s^{-1}, $\theta_0 \sim 1$, and $p \sim 10^{-3}$ cm s^{-1}, equation (7) leads to $L \sim 10$ m.

2.2. *U-shaped valley*

2.2.1. Geometry. We assume that the initial landscape is parabolic (figure 2(*a*)):

$$z = z_0 - \theta_0 x + \frac{y^2\theta_0}{2R}. \tag{8}$$

The lines of equal altitude correspond to

$$y^2 = 2R\left(x - \frac{z_0}{\theta_0}\right) \tag{9}$$

and the length R is their radius of curvature near the median line ($y = 0$). The local derivative $\mathrm{d}y/\mathrm{d}x$ along an isolevel line is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{R}{y}. \tag{10}$$

The flow lines (normal to the isolevel lines) are ruled by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y}{R} \tag{11}$$

$$y = a \exp\left(\frac{x_s - x}{R}\right) \tag{12}$$

where x_s corresponds to the starting point (at $y = a$).

The area A collected at level x inside a strip of small width y near the median is the hatched area of figure 2(*b*), and

$$A = a\left[x - R\ln\left(\frac{a}{y}\right) + R\right]. \tag{13}$$

Thus the collection factor (at small y) is

$$\frac{\mathrm{d}A}{\mathrm{d}l} \sim \frac{\mathrm{d}A}{\mathrm{d}y} = \frac{Ra}{y}. \tag{14}$$

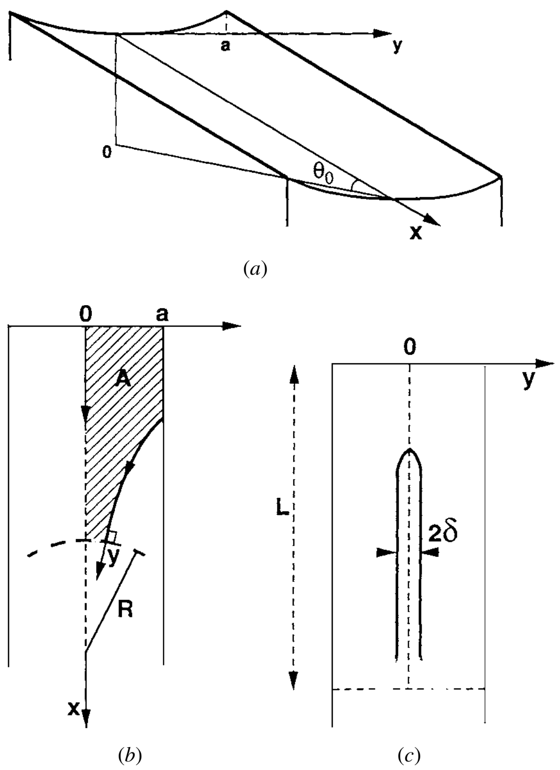

Figure 2. U-shaped valley: (*a*) the landscape, (*b*) isolevel lines (broken lines) and flow lines (arrows), and (*c*) aspect of the line of attack, separating uneroded regions (uphill) from eroded regions (downhill).

2.2.2. Line of attack. This is obtained by imposing $V = V_c$ in (5) and making use of (14). For small y, the slope is $\theta \cong \theta_0$. This leads to

$$y = \delta \equiv \frac{Ra}{L}. \tag{15}$$

Thus the line of attack is now *parallel to the median.* Equation (15) is the central result of this section. A few comments are useful here:

- the half width δ of the eroded region is proportional to the rainfall p and is a decreasing function of the threshold stress τ_c ($\delta \sim \tau_c^{-3/2}$) and
- the regimes of interest correspond to $y \ll a$ or $R \ll L$.

The distance x cannot be smaller than R: if $x < R$, the starting point is not on the lateral crest, but is at $x = 0$, and the collection factor is weaker. This ultimately leads to a shape for the line of attack which is qualitatively represented in figure 2(*c*).

3. Mature basins

Our aim now is to describe both erosion and sedimentation by an extension of the same ideas. Let us call $R(x, y, t)$ the local amount of moving solid (measured in terms of an equivalent height). To make things more palpable, we now write down a specific model for erosion and sedimentation:

$$\frac{\partial z}{\partial t} = -\frac{K}{V_s}(V^2 - V_e^2) + w\frac{R}{h} + U. \tag{16}$$

The first term describes erosion, and assumes that the rate is proportional to $\tau - \tau_c$. This linear law may be insufficient in practice, as pointed out in [5, ch 6]. It turns out, however that more general power laws lead to the same qualitative conclusions. Thus, for simplicity, we present our results with the linear model. The second term describes sedimentation, assumed to be proportional to the volume fraction of solid R/h in the flow. The last term is a very small uplift velocity. K is a material constant. It is dimensionless (and small), w has the dimensions of a velocity: it will in general depend on V, but for our purposes all these features are not essential.

Consider first the one-dimensional case where z depends only on the down slope coordinate x. The general aspect with uplift is shown in figure 3. There is a top region with no erosion, no sedimentation, and an uplift velocity U. This stops at a certain line of attack ($x = x_a(t)$). Below this point ($x > x_a$) we can have a steady-state solution with $\partial z/\partial t = 0$. We also have a book-keeping for the sediment, which in a steady state is produced at a rate U in all of the interval $x - x_a$

$$VR = U(x - x_a).$$

Comparing this with (4), we see that

$$\frac{R}{h} = \frac{x - x_a}{x}\frac{U}{p}$$

and (16), in a steady state, gives us

$$\frac{V^2 - V_c^2}{V_c} = K^{-1}U\left(1 + \frac{x - x_a}{x}\frac{W}{p}\right). \tag{17}$$

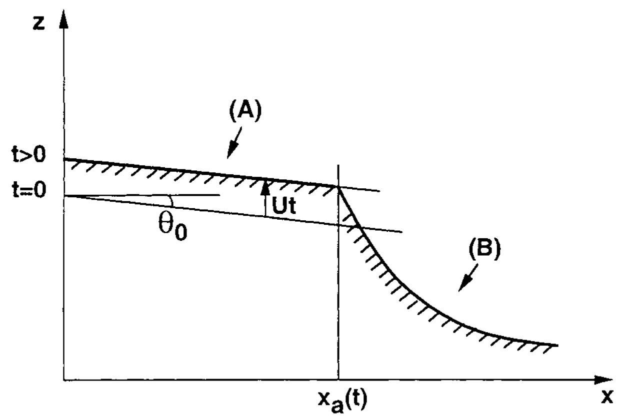

Figure 3. Predicted one-dimensional profiles for a mature basin with a slow uplift velocity U, (A) unperturbed region and (B) domain of erosion and sedimentation.

The crucial point is that U is geologically small: thus V is *necessarily close to* V_c. This remains true if we replace (16) by more realistic power laws.

This allows us to find immediately the steady-state profile: using (5) with $V = V_s$, we obtain

$$\theta \equiv \frac{-\mathrm{d}z}{\mathrm{d}x} = \theta_0\frac{L}{x}. \tag{18}$$

Thus the profile is expected to be logarithmic:

$$z(x) = z_1 - \theta_0 L \ln \frac{x}{L} \tag{19}$$

where z is a constant, which depends on the boundary conditions downstream. It is important to notice that the line of attack ($x = x_a$) moves slightly upward during time. On the line, we have

$$z = Ut + z_0 - \theta_0 x_a = z_1 - \theta_0 L \ln \frac{x_a}{L} \tag{20}$$

and this is an implicit equation for $x_a(t)$.

These considerations can be extended to more general basin shapes: x is replaced by $\mathrm{d}A/\mathrm{d}l$ and $x - x_a$ is replaced by $\mathrm{d}\tilde{A}/\mathrm{d}l$, where $\tilde{A}$ is the area collected between the line of attack and the level of observation.

4. Concluding remarks

(1) The main predictions from the model are: (a) the original position of the line of attack (equation (7) or (15)); (b) the steady-state profile, in the one-dimensional case, with a slope inversely proportional to the distance from the crest (equation (20)). The latter differs significantly from field observations (see [5, section 1.2.10]). This may mean that the weakly cohesive systems which we have in mind are very different from natural basins, but hopefully these laws could be compared to laboratory experiments. The solid material must be weak (low V_c). However, erosion should not be dominated by the direct impact of rain droplets. This is feasible is the droplet diameter is small; for $d = 1$ μm, the fall velocity of the droplets is of order 1 cm s^{-1}, and this is much smaller than V_c ($\sim$10 cm s^{-1}). Thus erosion by flow may be the leading process.

(2) An obvious question is related to the stability of the flows that we have described. Starting from an inclined plane as in section 2, can we have valleys forming spontaneously? The answer can be obtained from our discussion of U-type valleys: ultimately, *no linear instability is expected* in the region $x < L$, because to start erosion we need a concentration factor $\mathrm{d}A/\mathrm{d}l$, which is significantly larger than x, and this would impose a valley system of finite amplitude.

(3) Another open question, with the U-shaped valleys, concerns the evolution of the 'gully' (of initial width 2δ) which appears near the median line. We postpone the discussion for a later study.

(4) The mechanism of erosion initiation is definitely not the unique process responsible for channel-like patterns. However, in some simple systems, the deterministic models (initiated by Smith and Bretherton [4], Rodriguez-Iturbe and Rinaldo [5] and others) do lead to scaling predictions which are universal—i.e. independent of the detailed laws for erosion and sedimentation. The only crucial feature is the existence of a velocity threshold.

Acknowledgment

This text has greatly benefited from comments and criticisms by A Rinaldo.

Appendix. Exchanges between streamlets

The limits of the streamlet approximation appear clearly if we think of a flow on an inclined plane, with constant flux $V_0 h_0$ and constant velocity. Assume that in a small region near the starting point ($z = z_0$) the height h is increased $h = h_0 + h_1(y)$. Then the corresponding 'streamlet' will spread out. Ignoring capillary effects (gravity regime) we expect a transverse flow velocity:

$$V_y \sim \frac{-V_0}{\theta_0}\frac{\partial h_1}{\partial y} \tag{A1}$$

leading to a diffusion equation

$$\frac{\partial h_1}{\partial t} = -V_0\frac{\partial h_1}{\partial x} + \frac{-V_0 h_0}{\theta_0}\frac{\partial^2 h_1}{\partial y^2}. \tag{A2}$$

If your streamlet was infinitely narrow at the start, after a distance x downhill, it then reaches a width y_d of order

$$y_d \sim \left(\frac{h_0 x}{\theta_0}\right)^{1/2}. \tag{A3}$$

All our discussion of the convergence in a U-valley (section 2) assumed that the corresponding y_d is smaller than δ. For the U-valley problem, the uphill length of a flow line is of order R (rather than x) and thus $y_d \sim (h_0 R/\theta_0)^{1/2}$.

References

[1] Eagleson P S 1970 *Dynamic Hydrology* (New York: McGraw-Hill)

[2] Scheidegger A 1970 *Theoretical Geomorphology* (Berlin: Springer)

[3] Howard A D 1994 *Water Resour. Res.* **30** 2107–17

[4] Smith T and Bretherton F 1972 *Water Resour. Res.* **8** 1506–29

[5] Rodriguez-Iturbe I and Rinaldo A 1997 *Fractal River Basins* (Cambridge, UK: Cambridge University Press) (and references therein)

[6] Montgomery D and Dietrich W 1988 *Nature* **336** 232–4

[7] Montgomery D and Dietrich W 1992 *Science* **255** 826–9